ON LOG CANONICAL THRESHOLDS, II

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Abstract. We prove that the only accumulation points of the set $\mathcal{T}_3$ of all three-dimensional log canonical thresholds in the interval $[1/2, 1]$ are $1/2 + 1/n$, where $n \in \mathbb{Z}$, $n \geq 3$.

1. Introduction

In this paper we continue our study of the structure of the set $\mathcal{T}_3$ of all three-dimensional log canonical thresholds started in [10]. Notation and results of the Log Minimal Model Program [7] will be used freely.

Let X be a normal algebraic variety and let F be an effective integral non-zero $\mathbb{Q}$-Cartier divisor on X. Assume that X has at worst log canonical singularities. *The log canonical threshold* of (X, F) is defined by

$$c(X, F) = \sup \{c \mid (X, cF) \text{ is log canonical}\}.$$

For each $d \in \mathbb{Z}$, $d \geq 2$ define the following set $\mathcal{T}_d \subset [0, 1]$ by

$$\mathcal{T}_d := \left\{ c(X, F) \;\middle|\; \begin{array}{l} \dim X = d,\ X \text{ has only log canonical singu-} \\ \text{larities and } F \text{ is an effective non-zero Weil} \\ \mathbb{Q}\text{-Cartier divisor} \end{array} \right\}.$$

The above does not define $\mathcal{T}_1$ but it is naturally to put

$$\mathcal{T}_1 := \left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{N} \cup \{\infty\} \right\}.$$

The sets $\mathcal{T}_d$ have rather inductive nature: it is easy to show that $\mathcal{T}_{d-1} \subset \mathcal{T}_d$ and $\partial\mathcal{T}_d \supset \mathcal{T}_{d-1}$ (see [6, 8.21]), where $\partial\mathcal{T}$ is the set of all accumulation points of $\mathcal{T}$.

Conjecture 1.1 ([6])**.** *The accumulation set $\partial\mathcal{T}_d$ of $\mathcal{T}_d$ is precisely $\mathcal{T}_{d-1}$.*

This conjecture is the only one instance where the such a phenomena occurs. The similar behavior is expected for the fractional indices of log Fano varieties [11], [1], minimal log discrepancies [11], [13], [3], Kodaira energy [4] etc.

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In dimension two Conjecture 1.1 easily follows from explicit description of $\mathcal{T}_2$ [8]. In this paper we generalize the result of [10] and prove Conjecture 1.1 in dimension three for the interval $\left[\frac{1}{2}, 1\right]$:

Theorem 1.2.

$$\partial\mathcal{T}_3 \cap \left[\frac{1}{2}, 1\right] = \mathcal{T}_2 \cap \left[\frac{1}{2}, 1\right] = \left\{ \frac{1}{2} + \frac{1}{n} \quad \middle| \quad n \in \mathbb{Z}, n \geq 3 \right\}. \tag{1.1}$$

Note that (1.1) very similar to the corresponding results for log Del Pezzo surfaces [1]. Our proof based on inductive arguments and boundedness result [2]. As an intermediate result, we prove the following easy but very important fact:

Proposition 1.3. *Assume the LMMP in dimension d. Let $X \ni o$ be a d-dimensional $\mathbb{Q}$-factorial log terminal singularity* and let F be an (integral) Weil divisor on X. Let $c := c(X, F)$ be the log canonical threshold. Then one of the following holds:*

(i) $c \in \mathcal{T}_{d-1}$; *or*

(ii) $c \notin \mathcal{T}_{d-1}$ *and there is exactly one divisor S of the function field $\mathcal{K}(X)$ with discrepancy $a(S, cF) = -1$ (i.e., the pair (X, cF) is* exceptional *in the sense of* [12]*).*

Moreover in case (ii)*,* $\mathrm{Center}(S) = o$.

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2. Preliminary results

Notation. All varieties are assumed to be algebraic varieties defined over the field $\mathbb{C}$. A *log variety* (or a *log pair*) (X, D) is a normal quasiprojective variety X equipped with a *boundary* that is a $\mathbb{Q}$-divisor $D = \sum d_i D_i$ such that $0 \leq d_i \leq 1$ for all i. We use terminology, definitions and abbreviations of the Log Minimal Model Program [7]. Recall that $a(E, D)$ denotes the discrepancy of E with respect to D and

$$\begin{aligned} \mathrm{discr}(X, D) &= \inf_E \{a(E, D) \mid \mathrm{codim}\,\mathrm{Center}(E) \geq 2\}. \\ \mathrm{totaldiscr}(X, D) &= \inf_E \{a(E, D) \mid \mathrm{codim}\,\mathrm{Center}(E) \geq 1\}. \end{aligned}$$

Recall also our notation of [10]:

$$\begin{aligned} \Phi_{\mathbf{sm}} &= \left\{1 - \tfrac{1}{m} \mid m \in \mathbb{N} \cup \{\infty\}\right\}, \\ \Phi_{\mathbf{sm}}^{\alpha} &= \Phi_{\mathbf{sm}} \cup [\alpha, 1], \quad \text{for } \alpha \in [0, 1]. \end{aligned}$$

*By [10, Lemma 4.1] computing $\mathcal{T}_d$ we can consider only those singularities X which are $\mathbb{Q}$-factorial and log terminal.

Let Φ be any subset of $\mathbb{Q}$ and let $D = \sum D_i$ be a $\mathbb{Q}$-divisor. We write $D \in \Phi$ if $d_i \in \Phi$ for all i.

Lemma 2.1. *Fix a constant $N \in \mathbb{Z}$, $N \geq 6$. Let $\Lambda = \sum_{i=1}^{r} \lambda_i \Lambda_i$ be a boundary on $\mathbb{P}^1$ such that*
(i) $K_{\mathbb{P}^1} + \Lambda \equiv 0$;
(ii) $\Lambda \in \Phi_{\mathbf{sm}}^{\frac{1}{2}+\frac{1}{N}}$; *and*
(iii) $1 > \lambda_j > \frac{1}{2} + \frac{1}{N}$ *for some* j.
Then $\lambda_i \leq 1 - \frac{1}{N}$ for all i.

Proof. Clearly, $r = 3$ and $\lfloor \Lambda \rfloor = 0$. Assume that $\lambda_1 > 1 - \frac{1}{N}$. Then

$$1 < \lambda_2 + \lambda_3 < 1 + \frac{1}{N}.$$

Since $\lambda_i \geq \frac{1}{2}$, we have $\lambda_2, \lambda_3 < \frac{1}{2} + \frac{1}{N}$. Thus $\lambda_2 = \lambda_3 = \frac{1}{2}$ and $\lambda_1 = 1$, a contradiction. $\square$

Lemma 2.2. *Fix a constant $N \in \mathbb{N}$, $N \geq 6$. Let $(S \ni o, \Theta = \sum \vartheta_i \Theta_i)$ be a klt log surface germ with $\Theta \in \Phi_{\mathbf{sm}}^{\frac{1}{2}+\frac{1}{N}}$. Define the following boundary Ξ with* $\operatorname{Supp}(\Xi) = \operatorname{Supp}(\Theta)$:

$$\Xi := \sum \xi_i \Theta_i, \qquad \xi_i = \begin{cases} 1 & \text{if } \vartheta_i > 1 - \frac{1}{N}, \\ \vartheta_i & \text{otherwise.} \end{cases} \tag{2.2}$$

Then (S, Ξ) is lc.

Proof. If $\vartheta_i \leq 1 - \frac{1}{N}$ for all i, there is nothing to prove. Assume that $\Xi \neq \Theta$ and (S, Ξ) is not lc. Replacing Θ with $\Theta + \alpha(\Xi - \Theta)$, $\alpha > 0$, we may assume that (S, Θ) is lc but not klt (and $\lfloor \Theta \rfloor = 0$). Let $\mu \colon \overline{S} \to S$ be an *inductive blowup*† of the pair (S, Θ) (see [9, Prop. 5]) and let E be the exceptional divisor. By definition, E is irreducible, $a(E, \Theta) = -1$ and $(\overline{S}, E)$ is plt. Write

$$\mu^*(K_S + \Theta) = K_{\overline{S}} + E + \overline{\Theta},$$

where $\overline{\Theta}$ is the proper transform of Θ. Clearly, $\mu(E) \in \Theta_j$ with $\vartheta_j > 1 - \frac{1}{N}$.

2.3. By [10, Corollary 2.5], $\operatorname{Diff}_E(\overline{\Theta}) \in \Phi_{\mathbf{sm}}^{\frac{1}{2}+\frac{1}{N}}$. Pick a point $\overline{P} \in E \cap \overline{\Theta}_j$. Then $\overline{\Theta}_j$ is the only component of $\overline{\Theta}$, passing through $\overline{P}$ (see [10, Corollary 2.4]). Moreover, $(\overline{S}, E + \overline{\Theta}_j)$ is lc at $\overline{P}$ [10, Lemma 3.2]. Hence $(\overline{S}, E + \overline{\Theta})$ is plt at $\overline{P}$ and the coefficient λ' of $\operatorname{Diff}_E(\overline{\Theta})$ at $\overline{P}$ satisfies the inequality $1 - \frac{1}{N} < \lambda' < 1$. Therefore, $\Lambda := \operatorname{Diff}_E(\overline{\Theta})$

†In [9] such a μ was called *plt-blowup of the pair* (S, Θ).

satisfies conditions of Lemma 2.1. This gives us $\mathrm{Diff}_E(\overline{\Theta}) \in \left[1, \frac{1}{N}\right]$, a contradiction. $\square$

Lemma 2.4. *Let $(S \ni o, \Lambda = \sum \lambda_i \Lambda_i)$ be a log surface germ such that $\Lambda \in \left(1-\frac{1}{N}, 1\right]$. Assume that $\mathrm{discr}(S, \Lambda) \geq -1+\frac{1}{N}$ at o for $N \in \mathbb{Z}$, $N \geq 6$. Then $\sum \lambda_i \leq 2-\frac{1}{N}$. In particular, Λ has at most two components.*

Proof. For some $\Lambda' := \Lambda + t(\lceil \Lambda \rceil - \Lambda)$, $0 < t \leq 1$ the pair (S, Λ') is lc but not plt at o. By Lemma 2.2, we have $\Lambda' = \lceil \Lambda \rceil$, i.e., $(S, \lceil \Lambda \rceil)$ is lc. If Λ has only one component, there is nothing to prove. So, we may assume that Λ has exactly two components [7, Ch. 3]. Then near o we have

$$(S, \lceil \Lambda \rceil) \simeq_{\mathrm{an}} (\mathbb{C}^2, \{xy = 0\}, 0)/\mathbb{Z}_m(1, q),$$

where $m \in \mathbb{N}$ and $\gcd(m, q) = 1$. Take q so that $1 \leq q \leq m$. As in the proof of Lemma 3.3 in [10], considering the weighted blow up with weights $\frac{1}{m}(1, q)$ we get $\lambda_1 + \lambda_2 \leq 2 - \frac{1}{N}$. $\square$

3. Proof of Proposition 1.3. Corollaries

Notation and assumption as in Proposition 1.3. Let $f\colon Y \to X$ be an inductive blowup of the pair (X, cF) (see [9, Prop. 5]). Write

$$f^*(K_X + cF) = K_Y + cF_Y + S,$$

where F_Y is the proper transform of F and S is the (irreducible) exceptional divisor. By definition, (Y, S) is plt.

Assume that $c \notin \mathcal{T}_{d-1}$. If $f(S) \neq o$, then the pair (X, cF) is lc but not klt along $f(S)$. Taking the general hyperplane section we get $c \in \mathcal{T}_{d-1}$.

Hence $f(S) = o$. It is sufficient to show that $(Y, S + cF_Y)$ is plt (see [6, 3.10]). Assume the converse. Then there is an divisor $E \neq S$ of the field $\mathcal{K}(Y)$ such that $a(E, S + cF_Y) = -1$. Since (Y, S) is plt, $\mathrm{Center}_Y(E) \subset E \cap F_Y$.

Pick a point $P \in \mathrm{Center}_Y(E)$ and consider Y as a germ near P. Take the minimal $m \in \mathbb{N}$ such that $mS \sim 0$ near P and let

$$Y' := \mathrm{Spec}\Big(\bigoplus_{i=0}^{m-1} \mathcal{O}_Y(iS)\Big).$$

Then the projection $\varphi\colon Y' \to Y$ is an étale in codimension one $\mathbb{Z}_m$-covering. Put $P' := \varphi^{-1}(P)$, $F'_Y := \varphi^* F_Y$, and $S' := \varphi^* S$. Then (Y', S') is plt and $(Y, S' + cF'_Y)$ is lc but not plt near P' (see [12, §2]). Since S' is Cartier, $\mathrm{Diff}_{S'}(0) = 0$ (i.e., no codimension two components of $\mathrm{Sing}(Y')$ are contained in S'). By the Adjunction [7, Th. 17.6,

17.7] $(S', cF'_Y|_{S'})$ is lc but not klt near P'. Hence $c = c(S', F'_Y|_{S'})$ and $c \in \mathcal{T}_{d-1}$.

The Adjunction [7, 17.6] and [12, Cor. 3.10] gives us the following:

Corollary 3.1. *Let $c \in \mathcal{T}_d \setminus \mathcal{T}_{d-1}$. Assume that the LMMP in dimension d holds. Then there is a log pair (S, Θ) such that*

(i) *(S, Θ) is klt;*
(ii) *$K_S + \Theta \sim_{\mathbb{Q}} 0$;*
(iii) *$\Theta = \sum_i \vartheta_i \Theta_i$, where*

$$\vartheta_i = 1 - \frac{1}{m_i} + \frac{k_i c}{m_i}, \quad m_i \in \mathbb{N},\ k_i \in \mathbb{Z}_{\geq 0},\ k_i c < 1;$$

(iv) *$-\left(K_S + \sum_i (1 - \frac{1}{m_i})\Theta_i\right)$ is ample. In particular, $\sum k_i > 0$.*

Corollary 3.2. *Let $c \in \mathcal{T}_2 \setminus \mathcal{T}_1$. Then there are $m_i \in \mathbb{N}$, $k_i \in \mathbb{Z}_{\geq 0}$ such that*

$$k_i c < 1,\ \sum k_i > 0,\ \text{and} \quad \sum_i \left(1 - \frac{1}{m_i} + \frac{k_i c}{m_i}\right) = 2. \tag{3.3}$$

Moreover, allowing $k_i c = 1$ in (3.3), we get $c = \frac{1}{k_i} \in \mathcal{T}_1 \subset \mathcal{T}_2$. Conversely, if there are $m_i \in \mathbb{N}$, $k_i \in \mathbb{Z}_{\geq 0}$ satisfying (3.3), then $c \in \mathcal{T}_2$.

Proof. Apply Corollary 3.1. We obtain $S \simeq \mathbb{P}^1$ and $\deg \Theta = 2$. The inverse implication follows by [8]. $\square$

Corollary 3.3 ([8])**.** *Any $c \in \mathcal{T}_2 \cap (\frac{1}{2}, 1]$ has the following form*

$$\frac{1}{2} + \frac{1}{n}, \quad n \in \mathbb{Z},\ n \geq 2.$$

3.4. For $c \in [0, 1] \cap \mathbb{Q}$, let $\mathcal{LP}(c)$ be the class of all projective klt log surfaces (S, Θ) satisfying conditions (i)-(iv) of Corollary 3.1. Then

$$\mathcal{T}_3 \setminus \mathcal{T}_2 \subset \{c \mid \mathcal{LP}(c) \neq \varnothing\}.$$

Lemma 3.5. *Let c and (S, Θ) be as in Corollary 3.1 with $d = 3$. Assume that there is a contraction $g \colon S \to W$ onto a curve. Then all components Θ_i with $k_i > 0$ are vertical (i.e., $g(\Theta_i) \neq W$).*

Proof. Assume that there is a horizontal component Θ_i with $k_i > 0$. Let S_w be the general fiber. Then $S_w \simeq \mathbb{P}^1$ and by Adjunction we have equality (3.3):

$$\deg \Theta|_{S_w} = \sum_{\Theta_i \cap S_w \neq \varnothing} \left(1 - \frac{1}{m_i} + \frac{k_i c}{m_i}\right) = 2.$$

By our assumption, $\sum_{\Theta_i \cap S_w \neq \varnothing} k_i > 0$. Thus $c \in \mathcal{T}_2$, a contradiction. $\square$

Corollary 3.6. *Let $c \in \mathcal{T}_3 \setminus \mathcal{T}_2$. Then there is a log surface $(S, \Theta) \in \mathcal{LP}(c)$ with $\rho(S) = 1$.*

Proof. Denote

$$\Theta^c := \sum_{k_i > 0} \left(1 - \frac{1}{m_i} + \frac{k_i c}{m_i}\right) \Theta_i$$

and run $K_S + \Theta - \Theta^c$-MMP. Since $K_S + \Theta \equiv 0$, each time we contract an extremal ray R such that $R \cdot \Theta^c > 0$. Hence Θ^c is not contracted. By Lemma 3.5, at the end we obtain a model with $\rho = 1$. $\square$

4. Proof of the main theorem

In this section we prove Theorem 1.2.

Lemma 4.1. *For any $\epsilon > 0$ and $\frac{1}{2} > \xi > 0$ there exists a finite set $\mathcal{M}_{\xi,\epsilon} \subset [0, 1]$ such that $c \in \mathcal{M}_{\xi,\epsilon}$ whenever $c > \xi$ and there is $(S, \Theta) \in \mathcal{LP}(c)$ with*

$$\operatorname{totaldiscr}(S, \Theta) > -1 + \epsilon.$$

Proof. Since $\Theta \neq 0$, one can apply [2, Th. 6.9] to (S, Θ). This gives us that the family $\mathbf{S}$ of all such S is bounded. That is there is a family $\mathbf{S} \to \mathbf{H}$ such that every S is a fiber of $\mathbf{S} \to \mathbf{H}$. Therefore there is a polarization $\mathbf{L}$ on $\mathbf{S}$ giving us an embedding $\mathbf{S} \hookrightarrow \mathbb{P}$ over $\mathbf{H}$. This induces a very ample divisor L on each $S \in \mathbf{S}$. For all coefficients of Θ we have $\vartheta_i > \xi$. Then

$$L \cdot \sum_i \Theta_i < -\frac{1}{\xi} L \cdot K_S \leq \operatorname{Const}(\epsilon, \xi).$$

Hence the family of all $\sum \Theta_i$ is represented by a closed subscheme of $\mathbb{P}$ over $\mathbf{H}$. This shows that the pair $(S, \operatorname{Supp}(\Theta))$ is bounded. From the equality

$$L \cdot K_S + L \cdot \Theta = 0$$

we obtain the following linear equation in c:

$$L \cdot K_S + \sum_i \left(1 - \frac{1}{m_i} + \frac{k_i c}{m_i}\right) (L \cdot \Theta_i) = 0,$$

where

$$1 - \frac{1}{m_i} + \frac{k_i c}{m_i} \leq -\operatorname{totaldiscr}(S, \Theta) < 1 - \epsilon.$$

This gives us a finite number of possibilities for the m_i, k_i and c. $\square$

Lemma 4.2. *Fix constants $N \in \mathbb{Z}$, $N \geq 6$ and $0 < \epsilon < \frac{1}{N}$. Let $(S, \Theta = \sum \vartheta_i \Theta_i)$ be a klt log surface such that $\Theta \in \Phi_{\mathbf{sm}}^{\frac{1}{2}+\frac{1}{N}}$. Assume that there are at least two divisors of the function field $\mathcal{K}(S)$ such that*

$$a(\quad, \Theta) < -1 + \frac{1}{N} - \epsilon.$$

Then

$$\mathrm{totaldiscr}(S, \Theta) > -1 + \epsilon.$$

Proof. Let $\mu \colon \widetilde{S} \to S$ be the blowup of all divisors with discrepancies $a(\quad, \Theta) < -1 + \frac{1}{N}$ (see [7, Th. 17.10, 2.12.2]) and let $\widetilde{\Theta}$ be the crepant pullback of Θ:

$$K_{\widetilde{S}} + \widetilde{\Theta} = \mu^*(K_S + \Theta), \quad \mu_* \widetilde{\Theta} = \Theta.$$

Then $(\widetilde{S}, \widetilde{\Theta})$ satisfies conditions of Lemma 4.2. Moreover,

$$\mathrm{discr}(\widetilde{S}, \widetilde{\Theta}) \geq -1 + \frac{1}{N}.$$

Clearly,

$$\mathrm{totaldiscr}(S, \Theta) = \mathrm{totaldiscr}(\widetilde{S}, \widetilde{\Theta})$$

(see [6, 3.10]). Replace (S, Θ) with $(\widetilde{S}, \widetilde{\Theta})$. Up to permutations of the Θ_i we may assume that

$$\vartheta_1, \vartheta_2 > 1 - \frac{1}{N} + \epsilon.$$

Now it is sufficient to show that $\vartheta_i < 1 - \epsilon$ for all i. Consider the boundary Ξ with $\mathrm{Supp}(\Xi) = \mathrm{Supp}(\Theta)$ as in (2.2). Then $\lfloor \Xi \rfloor = \lceil \Xi - \Theta \rceil$. For a sufficiently small positive rational α, the $\mathbb{Q}$-divisor $\Theta - \alpha(\Xi - \Theta)$ is a boundary. It is clear that

$$K_S + \Theta - \alpha(\Xi - \Theta) \equiv -\alpha(\Xi - \Theta)$$

cannot be nef. By Lemma 2.2 the pair (S, Ξ) is lc.

Run $K_S + \Theta - \alpha(\Xi - \Theta)$-MMP. On each step we contract an extremal ray R such that

$$(K_S + \Xi) \cdot R = (\Xi - \Theta) \cdot R > 0.$$

4.3. We claim that none of the components of $\lfloor \Xi \rfloor$ is contracted. Indeed, assume that $\varphi \colon S \to S^o$ contracts $C \subset \lfloor \Xi \rfloor$. Take $\Theta' := \Theta + \beta C$ so that $\lfloor \Theta' \rfloor = C$ and $\Theta' \leq \Xi$. Since $(K_S + \Xi) \cdot C > 0$ and $(K_S + \Theta) \cdot C < 0$, there is a component, say Θ_0, of $\lfloor \Xi \rfloor$ meeting C. Further, take $\Theta'' := \Theta' + \gamma(\Xi - \Theta')$ so that $(K_S + \Theta'') \cdot C = 0$. Then $0 < \gamma < 1$. It is easy to see that $\Theta'' \in \Phi_{\mathbf{sm}}^{\frac{1}{2}+\frac{1}{N}}$ and $\lfloor \Theta'' \rfloor = C$. Note that (S, Θ'') is lc (because so is (S, Ξ)). As in the 2.3, we can apply

Lemma 2.1 to $\mathrm{Diff}_C(\Theta''-C)$ to derive a contradiction. This proves our claim.

4.4. By Lemma 2.2 the lc property of (S,Ξ) is preserved on each step. At the end of the MMP we get a birational model $(\overline{S},\overline{\Theta})$ with nonbirational extremal $\overline{\Xi}-\overline{\Theta}$-positive contraction $g\colon \overline{S}\to W$, where W is either a curve or a point.

4.4.1. *Subcase: W is a curve.* Then $\rho(\overline{S})=2$. Let $\overline{S}_w$ be the general fiber of g. Then $\mathrm{Diff}_{\overline{S}_w}(\Theta)$ satisfy conditions of Lemma 2.1. This yields a contradiction.

4.4.2. *Subcase: W is a point.* Then $\rho(\overline{S})=1$ and every two components of Θ intersects each other. By Lemma 2.4,

$$\vartheta_1\le 2-\frac{1}{N}-\vartheta_2<1-\epsilon.$$

Similarly, if $i\neq 1$ and the image of Θ_i on $\overline{S}$ is not a point, then

$$\vartheta_i\le 2-\frac{1}{N}-\vartheta_1<1-\epsilon.$$

But if Θ_i is contracted to a point on $\overline{S}$, then $\Theta\not\subset\lfloor\Xi\rfloor$. In this case, $\theta_i\le 1-\frac{1}{N}<1-\epsilon$. This proves our lemma. □

4.5. Now we are ready to prove Theorem 1.2. Assume that there is a sequence $c_n\in\mathcal{T}_3\cap[\frac{1}{2},1]$ such that $c_{n_1}\neq c_{n_2}$ for $n_1\neq n_2$ and $\lim c_n=c_\infty\notin\mathcal{T}_2$. Take constants $N\in\mathbb{N}$ and $\epsilon\in\mathbb{Q}$ so that

$$N\ge 6,\qquad \tfrac{1}{2}+\tfrac{1}{N}<c_\infty,\ \text{and}$$
$$0<\epsilon<\min\left\{c_\infty-\tfrac{1}{2}-\tfrac{1}{N},\ \tfrac{1}{N}\right\}.$$

By passing to a subsequence, we may assume that $c_n>\frac{1}{2}+\frac{1}{N}+\epsilon$ for all n. For every c_n we have the corresponding log surface $(S_n,\Theta_n)\in\mathcal{LP}(c_n)$ with $\rho(S_n)=1$ (see Corollaries 3.1 and 3.6). In particular, $\Theta_n\in\Phi_{\mathbf{sm}}^{\frac{1}{2}+\frac{1}{N}+\epsilon}$. Write $\Theta_n=\sum_i\vartheta_{n,i}\Theta_{n,i}$. By construction,

$$\vartheta_{n,i}=1-\frac{1}{m_{n,i}}+\frac{k_{n,i}c_n}{m_{n,i}},\quad k_{n,i}c_n<1,\quad \sum_i k_{n,i}>0. \tag{4.4}$$

If

$$\lim_{n\to\infty}\mathrm{totaldiscr}(S_n,\Theta_n)>-1,$$

we can take $\nu>0$ so that $\mathrm{totaldiscr}(S_n,\Theta_n)\ge -1+\nu$ for $n\gg 0$, then c_n belongs to a finite set $\mathcal{M}_{\frac{1}{2}+\frac{1}{N},\nu}$ by Lemma 4.1. This contradicts to

our choice of the sequence c_n. From now on we assume that

$$\lim_{n\to\infty} \operatorname{totaldiscr}(S_n, \Theta_n) = -1, \tag{4.5}$$

In particular,

$$\operatorname{totaldiscr}(S_n, \Theta_n) < -1 + \frac{1}{N} - \epsilon \text{ for all } n.$$

Assume that for $n \gg 0$ there are at least two divisors of the field $\mathcal{K}(S_n)$ with discrepancies $a(\quad, \Theta_n) < -1 + \frac{1}{N} - \epsilon$. Then (S_n, Θ_n) satisfies conditions of Lemma 4.2. Therefore

$$\operatorname{totaldiscr}(S_n, \Theta_n) > -1 + \epsilon,$$

This contradicts (4.5).

4.6. **Main case.** Finally we consider the case when for $n \gg 0$ there is exactly one divisor Γ_n with

$$\gamma_n := -a(\Gamma_n, \Theta_n) > 1 - \frac{1}{N} + \epsilon.$$

We construct a new birational model $(\overline{S}_n, \gamma_n\overline{\Gamma}_n + \overline{\Theta}_n)$ of (S_n, Θ_n) with $\rho(\overline{S}_n) = 1$ and such that the center of Γ_n on $\overline{S}_n$ is a curve.

4.6.1. If $\operatorname{Center}_{S_n}(\Gamma_n)$ is a curve, then $\Gamma_n = \Theta_{n,i}$ and $\gamma_n = \vartheta_{n,i}$ for some i. In this case we just put $\overline{S}_n := S_n$ and $\overline{\Theta}_n := \Theta_n - \gamma_n\Gamma_n$. Thus

$$\overline{\Theta}_n = \sum_i \overline{\vartheta}_{n,i}\overline{\Theta}_{n,i},$$

where $\overline{\Theta}_{n,i} := \Theta_{n,i}$ whenever $\Theta_{n,i} \neq \Gamma_n$ and

$$\overline{\vartheta}_{n,i} = \begin{cases} 0 & \text{if } \Theta_{n,i} = \Gamma_n, \\ \vartheta_{n,i} & \text{otherwise.} \end{cases}$$

4.6.2. If $\operatorname{Center}_{S_n}(\Gamma_n)$ is a point, we consider the blowup of this Γ_n: $\mu\colon \widetilde{S}_n \to S_n$ [7, Th. 17.10]. Clearly, $\rho(\widetilde{S}_n) = 2$. Write

$$K_{\widetilde{S}_n} + \gamma_n\Gamma_n + \widetilde{\Theta}_n = \mu^*\left(K_{S_n} + \Theta_n\right),$$

$$\widetilde{\Theta}_n = \sum \vartheta_i\widetilde{\Theta}_{n,i}, \quad \text{where } \mu_*\widetilde{\Theta}_{n,i} = \Theta_{n,i}.$$

By construction, $\vartheta_{n,i} \le 1 - \frac{1}{N} + \epsilon$. The divisor $K_{\widetilde{S}_n} + \widetilde{\Theta}_n \equiv -\gamma_n\Gamma_n$ cannot be nef. Therefore, there is a Γ_n-positive extremal contraction $\varphi\colon \widetilde{S}_n \to \overline{S}_n$, where $\rho(\overline{S}_n) = 1$. By Lemma 2.2, $(\widetilde{S}_n, \Gamma_n + \widetilde{\Theta}_n)$ is lc. If $\overline{S}_n$ is a curve, we derive a contradiction as in 4.4.1.

Therefore φ is birational. Put $\overline{\Theta}_n := \varphi_*\widetilde{\Theta}_n$, $\overline{\Theta}_{n,i} := \varphi_*\widetilde{\Theta}_{n,i}$, and $\overline{\Gamma}_n := \varphi_*\Gamma_n$. Then $(\overline{S}_n, \gamma_n\overline{\Gamma}_n + \overline{\Theta}_n)$ is klt and $K_{\overline{S}_n} + \gamma_n\overline{\Gamma}_n + \overline{\Theta}_n$ is numerically trivial. Again by Lemma 2.2, $(\overline{S}_n, \overline{\Gamma}_n + \overline{\Theta}_n)$ is lc.

Further,

$$\overline{\Theta}_n = \sum_i \overline{\vartheta}_{n,i} \overline{\Theta}_{n,i},$$

where

$$\overline{\vartheta}_{n,i} = \begin{cases} 0 & \text{if } \varphi(\widetilde{\Theta}_{n,i}) \text{ is a point,} \\ \vartheta_{n,i} & \text{otherwise.} \end{cases}$$

4.6.3. In both cases 4.6.1 and 4.6.2 we have

$$\overline{\vartheta}_{n,i} \le 1 - \frac{1}{N} + \epsilon. \tag{4.6}$$

As in the proof of Lemma 4.1, apply [2, Th. 6.9] to $(\overline{S}_n, \overline{\Theta}_n)$. We get that the family of all $(\overline{S}_n, \mathrm{Supp}(\overline{\Theta}_n + \overline{\Gamma}_n))$ is bounded. By passing to a subsequence we may assume that all the discrete invariants $(\overline{\Gamma}_n)^2$, $\overline{\Gamma}_n \cdot K_{\overline{S}_n}$, $\Theta_{n,i} \cdot K_{\overline{S}_n}$, $p_a(\overline{\Gamma}_n)$, and $K^2_{\overline{S}_n}$ do no depend on n. For short denote them by $\overline{\Gamma}^2$, $\overline{\Gamma} \cdot K_{\overline{S}}$, $\Theta_i \cdot K_{\overline{S}}$, $p_a(\overline{\Gamma})$, and $K^2_{\overline{S}}$, respectively.

From (4.6) by passing to a subsequence we may assume that all constants $m_{n,i}$ and $k_{n,i}$ in (4.4) also do not depend on n:

$$\overline{\vartheta}_{n,i} = 1 - \frac{1}{m_i} + \frac{k_i c_n}{m_i}.$$

By the Adjunction [7, Ch. 16],

$$K_{\overline{\Gamma}_n} + \mathrm{Diff}_{\overline{\Gamma}_n}\left(\overline{\Theta}_n\right) \equiv (1-\gamma_n)\overline{\Gamma}_n|_{\overline{\Gamma}_n},$$

where $\mathrm{Diff}_{\overline{\Gamma}_n}\left(\overline{\Theta}_n\right) \ge 0$. Since $(\overline{S}_n, \overline{\Gamma}_n + \overline{\Theta}_n)$ is lc, $\mathrm{Diff}_{\overline{\Gamma}_n}\left(\overline{\Theta}_n\right)$ is a boundary (see [7, Prop. 16.6]). The coefficients of $\mathrm{Diff}_{\overline{\Gamma}_n}(\overline{\Theta}_n)$ have the same form as the coefficients of Θ_n:

$$\mathrm{Diff}_{\overline{\Gamma}_n}(\Theta_n) = \sum_j \left(1 - \frac{1}{s_j} + \frac{r_j c_n}{s_j}\right) P_j,$$

where $n_j \in \mathbb{N}$, $r_j \in \mathbb{Z}_{\ge 0}$, and $r_j c_n \le 1$ (see [12, Lemma 4.2]). Thus

$$\sum_j \left(1 - \frac{1}{s_j} + \frac{r_j c_n}{s_j}\right) = 2 - 2p_a(\overline{\Gamma}) + (1-\gamma_n)\overline{\Gamma}^2. \tag{4.7}$$

Here $\overline{\Gamma}^2 > 0$, $1 - \frac{1}{N} + \epsilon < \gamma_n < 1$ and $p_a(\overline{\Gamma}) \in \mathbb{Z}_{\ge 0}$. If $r_j = 0$ for all j, then γ_n can be found from the equation

$$\sum_j \left(1 - \frac{1}{s_j}\right) = 2 - 2p_a(\overline{\Gamma}) + (1-\gamma_n)\overline{\Gamma}^2.$$

In this case, $\gamma := \gamma_n$ does not depend on n and $\gamma < 1$. Therefore,

$$\mathrm{totaldiscr}(S^n, \Theta^n) > -\gamma > -1.$$

This contradicts our assumption (4.5).

Assume that there is at least one component with $r_i = 1$. Passing to the limit as $n \to \infty$ in (4.7) we obtain

$$\sum_j \left(1 - \frac{1}{s_j} + \frac{r_j c_\infty}{s_j}\right) = 2 - 2p_a(\overline{\Gamma}) + (1 - \gamma_\infty)\overline{\Gamma}^2.$$

If $\gamma_\infty < 1$, then

$$\lim_{n\to\infty} \operatorname{totaldiscr}(S^n, \Theta^n) \geq \min\left\{-\gamma_\infty, \quad -1 + \frac{1}{N} - \epsilon\right\} > -1.$$

Again we have a contradiction with (4.5). Hence $\gamma_\infty = 1$ and

$$0 < \sum_j \left(1 - \frac{1}{s_j} + \frac{r_j c_\infty}{s_j}\right) = 2 - 2p_a(\overline{\Gamma}).$$

This gives us that $p_a(\overline{\Gamma})$. By Lemma 3.1, $c_\infty \in \mathcal{T}_2$. Theorem 1.2 is proved.

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